Universal slow growth of entanglement in interacting strongly disordered systems

Maksym Serbyn,[1] Z. Papić,[2] and Dmitry A. Abanin[3,4]

[1] *Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02138, USA*
[2] *Department of Electrical Engineering, Princeton University, Princeton, NJ 08544, USA*
[3] *Perimeter Institute for Theoretical Physics, Waterloo, ON N2L 2Y5, Canada*
[4] *Institute for Quantum Computing, Waterloo, ON N2L 3G1, Canada*

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Recent numerical work by Bardarson *et. al.* [Phys. Rev. Lett. **109**, 017202 (2012)] revealed a slow, logarithmic in time, growth of entanglement entropy for initial product states in a putative many-body localized phase. We show that this surprising phenomenon results from the dephasing due to exponentially small interaction-induced corrections to the eigenenergies of different states. For weak interactions, we find that the entanglement entropy grows as $\xi \ln(Vt/\hbar)$, where V is the interaction strength, and ξ is the single-particle localization length. The saturated value of the entanglement entropy at long times is determined by the participation ratios of the initial state over the eigenstates of the subsystem. The proposed mechanism is illustrated with numerical simulations of small systems. Our work shows that the logarithmic entanglement growth is a universal phenomenon characteristic of the many-body localized phase in any number of spatial dimensions, and reveals a broad hierarchy of dephasing time scales present in such a phase.

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Introduction. While it is well-known that arbitrarily weak disorder localizes all single-particle quantum-mechanical states in 1 and 2 dimensions, the effect of disorder potential on the states of interacting many-body systems largely remains an open problem. Works [2, 3] conjectured that localization in a system of interacting electrons survives in the presence of weak interactions. When the strength of the interactions is increased, at some critical value a transition to the delocalized phase ("many-body localization transition") takes place. The existence of such a transition was further supported by numerical simulations of small systems [4–8, 12–16].

An important challenge is to understand the physical properties of the localized phase of interacting particles. Recent numerical work [1] (also see Ref. [5]) revealed that even very weak interactions dramatically change the growth of entanglement of non-equilibrium many-body states. Authors of Ref. [1] studied the time evolution of product states in a one-dimensional disordered XXZ spin chain, which can be reformulated in terms of interacting fermions in a random potential (the same model was also considered by other authors in the context of the many-body localization [8]). In the absence of interactions, such states maintain a low degree of entanglement upon evolution, and the entanglement entropy obeys an area law. In contrast, in the presence of interactions the states showed a slow, logarithmic in time, growth of entanglement entropy (here and below we use "entanglement" and "entanglement entropy" interchangeably). The saturated value of the entanglement entropy was found to vary approximately linearly with system size, and remained well below the maximum possible entanglement entropy of the system [1, 9]

In this paper, we identify a mechanism that underlies the logarithmic growth of entanglement in interacting localized states. The key observation is that although very weak interactions affect the many-body eigenstates of a non-interacting Hamiltonian only weakly, they do change the energies of these states. It is these small interaction-induced corrections to the energy which lead to the dephasing between different eigenstates on long time scales. We argue that this gives rise to a logarithmic growth of entanglement with time for a broad class of initial states that are a product of states in the two subsystems, including, but not limited to the initial states considered in Ref. [1].

For weak interactions, our mechanism leads to the following predictions regarding entanglement growth as a function of system's parameters: (i) entanglement entropy grows as $S_{\rm ent}(t) \propto \xi \log(Vt/\hbar)$, where V is the interaction strength and ξ is the single-particle localization length; (ii) the saturation value of $S_{\rm ent}$ is of the order of "diagonal entropy" $S_{\rm diag}$ [17] of a subsystem for a given initial state. Diagonal entropy is determined by the participation ratios of initial states in the basis of single-particle eigenstates. We also illustrate these predictions with numerical simulations of small systems, in particular by constructing examples of initial states for which the saturated entanglement is equal to the diagonal entropy.

Model. Without loss of generality, we consider a 1D lattice model of fermions with on-site disorder and nearest-neighbor interactions,

$$H = t\sum_{\langle ij\rangle} a_i^\dagger a_j + \sum_i W_i a_i^\dagger a_i + V\sum_{\langle ij\rangle} \hat{n}_i\hat{n}_j,\ \hat{n}_i = a_i^\dagger a_i, \tag{1}$$

where $i, j = 1, ...N$, and $\langle ij\rangle$ denotes nearest neighbors. This model is equivalent to the XXZ spin chain with a random field along the z-direction considered in Ref. [1]. From our discussion below, it will become clear that the

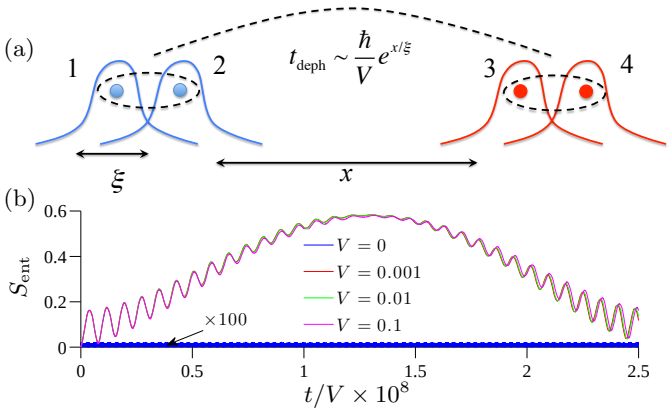

FIG. 1. (Color online) (a) Generation of entanglement for the case of a single particle in each half of the system prepared in a superposition of two eigenstates. Exponentially small overlap of the orbitals leads to the dephasing time growing exponentially with distance between particles. (b) Entanglement entropy as a function of time for a given realization of disorder and different interaction strengths. When $V = 0$, $S_{\rm ent} \sim 10^{-4}$ and remains small for all times (for the plot $t_{\max} = 2.5 \cdot 10^{11}$). For non-zero interactions, $S_{\rm ent}(t)$ collapse on a single curve when time is scaled by $1/V$. System size is $L = 10$ sites, disorder strength is $W = 6$. Initially, particles are in equal superposition of two sites at the opposite edges of the system [see panel (a)]. As these states are not the exact eigenstates, the maximum value of $S_{\rm ent}$ is slightly below $\ln 2 \approx 0.69$.

logarithmic growth of entanglement in interacting localized systems is a robust phenomenon which does not depend on the microscopic details, nor on the dimensionality of the system.

Throughout the paper, we focus mostly on the regime of weak interactions, for which the logarithmic growth of entanglement found in [1] is perhaps the most striking. In the absence of interactions, $V = 0$, disorder localizes the single-particle states, with localization length ξ. The many-body eigenstates are simply states in which a certain number of single-particle orbitals is occupied. Interactions that are much weaker compared to the typical level spacing on the scale ξ do not significantly modify the many-body eigenstates. We have explicitly verified this statement for small systems, and assume it holds in general. However, even though the eigenstates are not strongly affected by the interactions, their energies are modified. Interactions between remote particles yield a contribution to the energy that is exponentially small in the ratio x/ξ, where x is the distance between particles. Effectively, if we fix the positions of all particles, except for a pair of particles which are situated at a distance $x \gg \xi$ away from each other, the interaction energy of this pair is of the order $Ve^{-x/\xi}$, and the corresponding dephasing time is $t_{\rm deph} \sim \hbar e^{x/\xi}/V$. This gives rise to a hierarchy of dephasing time scales present in the problem, ranging from the fastest $t_{\min} = \hbar/V$ to the slowest $t_{\max} = t_{\min} e^{L/\xi}$, where L is the system size.

Generally, the product initial states considered in Ref. [1], as well as the initial states of other kinds considered below, are a superposition of many eigenstates. The interactions introduce a slow dephasing between different states, and effectively generate entanglement between different remote parts of the system. A subsystem of size x becomes nearly maximally entangled with the rest of the system after an exponentially long time $t_{\rm deph}(x) \sim \hbar e^{x/\xi}/V$. Thus, if one considers a bipartite division of the system, the entanglement entropy will increase logarithmically in time, as more and more degrees of freedom in the two subsystems become entangled with each other.

Two particles. Let us start with a simple example which demonstrates that the slow growth of entanglement occurs for just two particles. More specifically, consider two distant particles prepared in an equal-weight superposition of two neighboring localized orbitals, $|\Psi_0\rangle = \frac{c_1^\dagger + c_2^\dagger}{\sqrt{2}} \frac{c_3^\dagger + c_4^\dagger}{\sqrt{2}} |0\rangle$, where $c_i^\dagger$ creates an eigenstate localized near site i. We assume that the distance between wave functions 1,2 and 3,4 is large, $x \gg \xi$. This situation is depicted in Fig. 1.

In the absence of interactions, no entanglement would be generated during time evolution. Interactions change this picture by introducing a correction to the energy of the state $|\alpha\beta\rangle = c_\alpha^\dagger c_\beta^\dagger |0\rangle$, where $\alpha = 1, 2$, $\beta = 3, 4$. In the leading order of perturbation theory, the energy of this state is given by

$$E_{\alpha\beta} = \varepsilon_\alpha + \varepsilon_\beta + \delta E_{\alpha\beta}, \ \delta E_{\alpha\beta} = C_{\alpha\beta} V e^{-x/\xi}, \qquad (2)$$

where $\varepsilon_\alpha, \varepsilon_\beta$ are the single-particle energies, the last term is introduced by the interactions, and $C_{\alpha\beta}$ is a constant which depends only algebraically on x.

The time-evolved state is given by

$$|\Psi(t)\rangle = \frac{1}{2} \sum_{\alpha=1}^{2} \sum_{\beta=3}^{4} e^{-iE_{\alpha\beta}t} |\alpha\beta\rangle, \qquad (3)$$

and the reduced density matrix for the first particle reads

$$\hat{\rho}_L = \frac{1}{2} \begin{pmatrix} 1 & F(t)/2 \\ F^*(t)/2 & 1 \end{pmatrix}, \qquad (4)$$

where $F(t) = e^{-i\Omega t}(1 + e^{-i\delta\Omega t})$, $\delta\Omega = \delta E_{14} - \delta E_{24} - \delta E_{13} + \delta E_{23}$, and $\Omega = \varepsilon_1 - \varepsilon_2 + \delta E_{13} - \delta E_{23}$. The eigenstates of $\hat{\rho}_L$ therefore oscillate with a very long period $T = 2\pi/\delta\Omega \sim (\hbar/V) e^{x/\xi}$. At times $t = (2n+1)\pi/\delta\Omega$, the off-diagonal matrix elements vanish, and eigenstates become equal to $1/2$. At these times, the two particles become maximally entangled with $S_{\rm ent} = \ln 2$. Fig. 1 demonstrates that even weak interactions lead to the entanglement of the order of $S_{\rm ent} \approx \ln 2$, and the rate of entanglement change is inversely proportional to the interaction strength. Note that no disorder or time averaging is used.

General case. Turning to the general many-body case, let us divide the system into two parts, $\mathcal{L}$ and $\mathcal{R}$, labeling the single-particle orbitals that are localized dominantly in the left part by index α_n, and those residing in the right part by β_n. There may be some ambiguity for the state residing near the boundary between $\mathcal{L}$ and $\mathcal{R}$, but we will be interested in systems of size $L \gg \xi$, for which the boundary effects are not very important.

We consider initial states that are products of some superposition of states with a definite number of particles in $\mathcal{L}$ and $\mathcal{R}$:

$$|\Psi(t=0)\rangle = \sum_{\{\alpha\}\in\mathcal{L}} A_{\{\alpha\}}|\alpha_1...\alpha_K\rangle \times \sum_{\{\beta\}\in\mathcal{R}} B_{\{\beta\}}|\beta_1...\beta_M\rangle. \tag{5}$$

Coefficients A, B are chosen such that Ψ is normalized.

Neglecting the change to the eigenstate due to interactions, the reduced density matrix for $\mathcal{L}$ after time evolution reads

$$\hat{\rho}_{\mathcal{L}} = \sum_{\alpha,\alpha'} \rho_{\alpha\alpha'}|\alpha\rangle\langle\alpha'|, \tag{6}$$

$$\rho_{\alpha\alpha'} = A_\alpha A^*_{\alpha'} \sum_\beta |B_\beta|^2 \exp\left[i(E_{\alpha'\beta} - E_{\alpha\beta})t\right], \tag{7}$$

where we used a short-hand notation $\alpha \equiv \{\alpha\}, \beta \equiv \{\beta\}$. It is convenient to define $A_\alpha(t) = A_\alpha e^{-iE_\alpha t}$, where E_α is the energy of $|\alpha\rangle$ state for the isolated $\mathcal{L}$ subsystem. Assuming that $|\alpha\rangle \times |\beta\rangle$ remains an eigenstate (this may not be true near the boundary, but the boundary effect is not important for entanglement growth, at least in large systems), the above equation, written in terms of coefficients $A_\alpha(t)$, preserves the same form, except the energies $E_{\alpha\beta}$ should be substituted by the interaction energy $\delta E_{\alpha\beta}$ between particles in $\mathcal{L}$ and $\mathcal{R}$ subsystems. For particles that reside far away from the boundary, this correction can be calculated in perturbation theory.

The energy difference $\delta E_{\alpha'\beta} - \delta E_{\alpha\beta}$ that enters the off-diagonal elements of the reduced density matrix, to the leading order, is proportional to $Ve^{-x/\xi}$. Here x is the minimum distance between a particle in $\mathcal{L}$, the position of which is different in states α and α', and the particles in $\mathcal{R}$. However, it also contains many smaller contributions, which arise due to the interaction between more distant pairs of particles. Thus, the off-diagonal elements oscillate at a number of very different, incommensurate frequencies.

The interaction energy leads to dephasing, which decreases the off-diagonal elements of the density matrix, thus generating entanglement. Effectively, at times $t(x) \sim t_0 e^{x/\xi}$ the degrees of freedom within a distance $x(t) \sim \log(t/t_0)$ from the boundary between $\mathcal{L}$ and $\mathcal{R}$ are affected by dephasing, while states that differ only in the positions of particles further away from the boundary are still phase-coherent. This generates the entanglement entropy given by

$$S_{\rm ent}(t) = CS_{\rm diag},\ S_{\rm diag} = -\sum P_i(x)\ln P_i(x), \tag{8}$$

where $P_i(x)$ are the probabilities of different states $|\alpha\rangle$ in a segment of size x, calculated from the wave function of the initial state. Quantity $S_{\rm diag}$ is the diagonal entropy – a maximum achievable entropy for a given initial state, assuming that interactions do not change the eigenstates. $S_{\rm ent}$ is expected to be smaller than $S_{\rm diag}$ by a factor $C \lesssim 1$; the precise value of this pre-factor is non-universal, and depends on the preparation of the initial state. In the long-time limit, assuming that $\mathcal{R} \gg \mathcal{L}$ and the initial state is a superposition of many different states, we expect the off-diagonal elements to become very small such that the entanglement entropy approaches its maximum value with $C \to 1$.

Since for initial product states $S_{\rm diag}$ is proportional to the subsystem size, entanglement grows logarithmically:

$$S_{\rm ent}(t) \propto \xi \log(Vt/\hbar). \tag{9}$$

We emphasize that our argument does not rely on averaging, and therefore entanglement grows according to Eq.(9) even for a single disorder realization, and even for relatively small systems.

Numerical simulations. In order to illustrate the above mechanism, and to explore the growth of entanglement for different initial states, we performed numerical simulations of finite systems. We use XXZ spin chain as an effective description of fermions in 1D, with a random magnetic field uniformly distributed in the interval $[-W, W]$. Hopping, described by exchange $J_\perp$, is set to one, whereas the interaction strength is controlled by $J_z \equiv V$. In the many-particle simulations, the Hilbert space is restricted to the total $S_z = 0$ sector. Using exact diagonalization for systems up to 14 spins, we compute the time evolution of various initial states of the system. This allows us to obtain the entanglement entropy $S_{\rm ent}(t)$ between two halves of the system as a function of time (even number of sites is considered). We study its average $\bar{S}_{\rm ent}(t)$ over different initial states belonging to the same class, and over different realizations of disorder.

First, similar to Ref. [1], we consider a class of product states of up or down spin on every site ("up-down" states). Results for $\bar{S}_{\rm ent}(t)$ for a system of $L = 12$ sites and disorder $W = 5$ are shown in Fig. 2(a). After a rapid increase of entropy during time $t \sim 1/J_\perp$ due to diffusive transport on a scale smaller than the localization length, $\bar{S}_{\rm ent}(t)$ saturates for a non-interacting system. In the presence of even weak interactions, $\bar{S}_{\rm ent}(t)$ continues to grow further. The growth rate is found to be proportional to $\ln(Vt/\hbar)$ [Fig. 2(b)], in full agreement with our analysis above.

The saturation value of the entanglement, $\bar{S}_{\rm ent}(\infty)$, does not vary appreciably with the interaction strength when interactions are weak. This further supports the conclusion that weak interactions only weakly alter the eigenstates of the system. For fixed $V = 0.01$, $\bar{S}_{\rm ent}(\infty)$ decreases with disorder W [Fig. 2(c)]: for stronger disorder, the initial up-down state becomes increasingly more

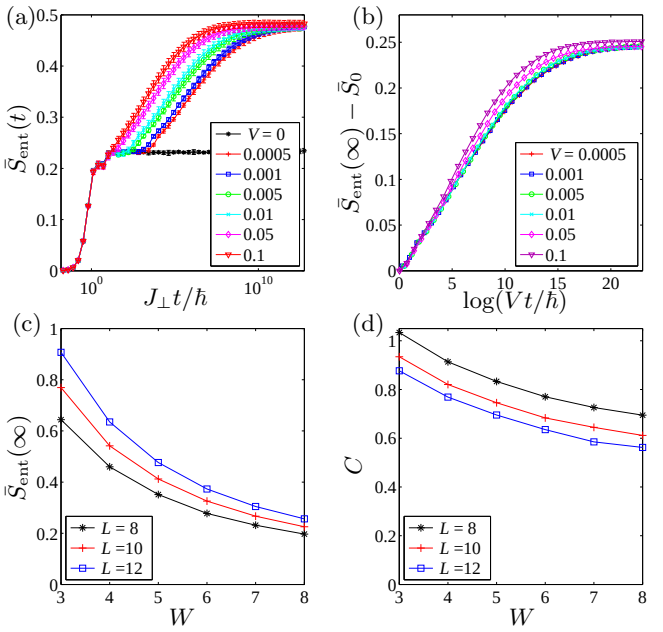

FIG. 2. (Color online) (a) Averaged entanglement entropy of a random "up-down" state as a function of time shows a characteristic logarithmic growth with time on long time scales. (b) $\bar{S}_{\rm ent} - \bar{S}_0$ collapses onto a single curve as a function of $\ln(Vt/\hbar)$. $\bar{S}_0$ is the saturation entropy for a non-interacting system. System size $L = 12$, $W = 5$. (c) Saturated entanglement for $V = 0.01$ decreases as a function of disorder strength. (d) Ratio $C = \bar{S}_{\rm ent}(\infty)/\bar{S}_{\rm diag}$ shows a slower decrease with W. Error bars (if not shown) are approximately equal to the size of symbols in each plot, with the exception of panel (b), where they are omitted for clarity.

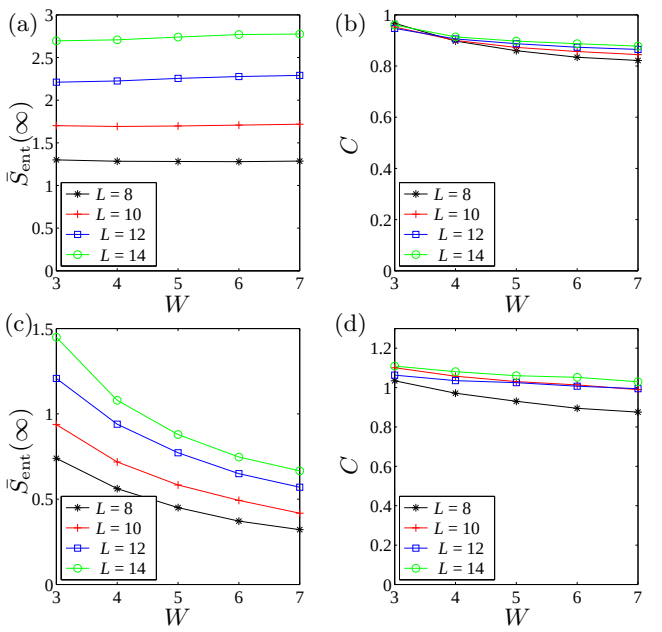

FIG. 3. (Color online) Saturation entropy as a function of disorder W for a strongly entangled state (a), and a product of a strongly entangled state and an "up-down" state (c). In the former case, the saturated entropy is approximately independent of W, while in the latter it decreases as W is increased. The ratio between saturation and diagonal entropy for two kinds of initial states, (b) and (d). In the second case, this ratio tends to one for larger system sizes as W is increased. Interaction strength is set to $V = 0.01$. Error bars (if not shown) are approximately equal to the size of symbols in each plot.

similar to an eigenstate of the system, and its diagonal entropy diminishes.

We compare the saturated entanglement to $S_{\rm diag}$, calculated using the values of $P_i(L/2)$ for the *initial state* of the $\mathcal{L}$ subsystem. The $P_i(L/2)$ are obtained from the density matrix, using the eigenstates of the interacting Hamiltonian restricted to $\mathcal{L}$. In this sense, while $\bar{S}_{\rm ent}(\infty)$ is determined from the time evolution of the system, $S_{\rm diag}$ is solely the property of the initial state.

To interpret the dependence of the ratio $C = \bar{S}_{\rm ent}(\infty)/\bar{S}_{\rm diag}$ on system size and disorder, Fig. 2(d), we must take into account two additional effects important for small systems: (i) the diffusion of particles across the entanglement cut; and (ii) the inefficiency of decoherence when the number of terms in Eq. (5) is small or when $\mathcal{L}$ and $\mathcal{R}$ are of equal size. These effects counteract each other, as diffusion leads to an additional contribution to the entanglement not captured by (8). On the other hand, inefficient decoherence leads to incomplete dephasing, and decreases the saturated entanglement compared to $S_{\rm diag}$. The positive contribution from (i) is suppressed for larger systems or smaller localization lengths. The effect of (ii) depends on the initial state. For the "up-down" states in the localized phase, the participation ratio is on order of unity, and the effect of (ii) is very pronounced. Thus, C is smaller than one, and it decreases with stronger disorder or system size [Fig. 2(d)].

Next, we consider another kind of initial states, which have larger participation ratios. The initial state of $\mathcal{R}$ and $\mathcal{L}$ subsystem is chosen as a projection to $S_z = 0$ sector of the product state of $(|\uparrow\rangle \pm |\downarrow\rangle)/\sqrt{2}$ per each site, with $\pm$ signs chosen at random. Spins within each subsystem are therefore strongly entangled, but there is no entanglement between the subsystems at $t = 0$. We found that the logarithmic entanglement growth occurs also in this case, but the saturated entropy, Fig. 3(a), is larger compared to the previous case, and varies weakly with disorder. Ratio of saturated to diagonal entropy, Fig. 3(b), now scales to one when the system size is increased, contrary to the "up-down" initial states. For this type of initial state, due to larger values of $\bar{S}_{\rm ent}(\infty)$, boundary diffusion contribution to entanglement is less important; also, the superposition of a large number of eigenstates in each half of the system makes decoherence more efficient, thus r is closer to one.

Finally, we construct an example where $S_{\rm ent}$ reaches

$S_{\rm diag}$. We take a product of the "up-down" state in $\mathcal{L}$, and the strongly entangled state in $\mathcal{R}$. To further suppress the diffusion, we require the two sites adjacent to the entanglement cut to be always in the $|\downarrow\rangle$ state. $\bar{S}_{\rm ent}(\infty)$ displays the behavior similar to the "up-down" case, Fig. 3(c), but is larger due to the more effective dephasing. Remarkably, Fig. 3(d) demonstrates that for larger system sizes, saturation and diagonal entropies become equal, in agreement with the above analysis.

Discussion. To summarize, we presented a mechanism for the logarithmic growth of entanglement in a model undergoing a many-body localization transition. We also established the laws that govern the entanglement growth, and tested them in numerical simulations for different initial states.

Although we focused on the limit of weak interactions, in which the eigenstates are similar to those of a non-interacting model, we expect our conclusions to hold also for stronger interactions which locally do modify the eigenstates. In this case, the saturated value of the entanglement is expected to be determined by the participation ratios of the initial state in the basis of interacting subsystem's eigenstates. Furthermore, our conclusions are expected to apply to localized interacting systems in any number of spatial dimensions

Our work also points to an exponentially broad distribution of dephasing time scales present in a many-body localized system. This has interesting implications for coherent dynamics of a small quantum system interacting with a many-body localized environment. For simplicity, let us consider a situation where the small quantum system is simply one of the particles that constitute the system, prepared in a superposition of some single-particle orbitals. Then, if a region of size L_* surrounding this particle is prepared in an eigenstate of the Hamiltonian (while the rest of the system is presumably in a mixed state), the dephasing time of the particle will be proportional to $\exp(L_*/\xi)$ and therefore exponentially long. This suggests that in certain systems it might be possible to reduce decoherence by introducing strong disorder in the environment.

Finally, we note that recently Vosk and Altman [18] considered an XXZ model with random exchange interaction, but without random field. For a special initial state, they developed a strong-disorder renormalization group (RG) procedure, and computed entanglement entropy, finding that it grows as a power of $\log t$. The difference from our result stems from the fact that the state considered in Ref. [18] was critical, however, the basic underlying mechanism – dephasing due to exponentially weak interactions between remote spins – is qualitatively similar.

Prior to submission, we became aware of a related independent work by Huse and Oganesyan [19]. Using a phenomenological description of the many body localized phase, they also predicted a logarithmic growth of entanglement with time. Our results agree insofar as they overlap.

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[1] J. H. Bardarson, F. Pollmann, and J. E. Moore, Phys. Rev. Lett. **109**, 017202 (2012).
[2] D. Basko, I. Aleiner, and B. Altshuler, Annals of Physics **321**, 1126 (2006), ISSN 0003-4916.
[3] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, Phys. Rev. Lett. **95**, 206603 (2005).
[4] V. Oganesyan and D. A. Huse, Phys. Rev. B **75**, 155111 (2007).
[5] M. Žnidarič, T. c. v. Prosen, and P. Prelovšek, Phys. Rev. B **77**, 064426 (2008).
[6] C. Monthus and T. Garel, Phys. Rev. B **81**, 134202 (2010).
[7] T. C. Berkelbach and D. R. Reichman, Phys. Rev. B **81**, 224429 (2010).
[8] A. Pal and D. A. Huse, Phys. Rev. B **82**, 174411 (2010).
[9] It is also worth noting that the slow growth of entanglement has been observed in random Heisenberg [10] and transverse-field Ising chains [11].
[10] G. D. Chiara, S. Montangero, P. Calabrese, and R. Fazio, Journal of Statistical Mechanics: Theory and Experiment **2006**, P03001 (2006).
[11] F. Iglói, Z. Szatmári, and Y.-C. Lin, Phys. Rev. B **85**, 094417 (2012).
[12] C. Gogolin, M. P. Müller, and J. Eisert, Phys. Rev. Lett. **106**, 040401 (2011).
[13] E. Canovi, D. Rossini, R. Fazio, G. E. Santoro, and A. Silva, Phys. Rev. B **83**, 094431 (2011).
[14] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Phys. Rev. Lett. **98**, 050405 (2007).
[15] F. Buccheri, A. De Luca, and A. Scardicchio, Phys. Rev. B **84**, 094203 (2011).
[16] A. De Luca and A. Scardicchio, Europhys. Lett. **101**, 37003 (2013).
[17] A. Polkovnikov, Annals of Physics **326**, 486 (2011).
[18] R. Vosk and E. Altman, Phys. Rev. Lett. **110**, 067204 (2013).
[19] D. A. Huse and V. Oganesyan, in preparation.